NONRADIAL SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION IN TWO DIMENSIONS

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Abstract : We establish existence of an infinite family of exponentially-decaying non-radial C^2 solutions to the equation $\Delta u + f(u) = 0$ on R^2 for a large class of nonlinearities f. These solutions have the form $u(r,\theta) = e^{im\theta}w(r)$, where r and θ are polar coordinates, m is an integer, and $w : [0,\infty) \to R$ is exponentially decreasing far from the origin. We prove there is a solution with each prescribed number of nodes.

1. INTRODUCTION

We consider the semilinear elliptic equation $\Delta u + f(u) = 0$, where the nonlinearity $f : C \to C$ has the property that $f(se^{i\psi}) = f(s)e^{i\psi}$ for all real s and ψ. The behavior of such a function is determined by its restriction to real arguments, and henceforth we refer only to the restriction of f to the real axis, which is necessarily odd. We look for C^2 solutions $u : R^N \to C$ such that $u(x) \to 0$ as $|x| \to \infty$. Interest in these solutions stems from their role as the spatial profiles of localized standing-wave solutions to nonlinear evolution equations, including the nonlinear Schrödinger and nonlinear wave equations. The set of spherically-symmetric ("radial") solutions has been extensively studied (see [1] - [4], [6] - [12]).

In this paper we use ordinary-differential-equation arguments to establish the existence of solutions that do not have rotational symmetry, in the case of $N = 2$ spatial dimensions. We make use of an ansatz due to P.-L. Lions ([8]) to reduce the study of the partial differential equation to that of an ordinary differential equation. Specifically, we look for solutions u of the form $u(r,\theta) = e^{im\theta}w(r)$, where r and θ are polar coordinates for R^2, m is a nonzero integer, and $w : [0,\infty) \to R$. Substituting this ansatz into the elliptic equation for u yields the ordinary differential equation $w'' + \frac{1}{r}w' - \frac{m^2}{r^2}w + f(w) = 0$ for w. Without loss of generality, we henceforth assume that m is a positive integer.

A method based on variational arguments for proving existence of solutions obeying this ansatz (and higher-dimensional generalizations) is outlined in [8]. In [7] solutions are explicitly computed for a piecewise-linear nonlinearity f. Here we use shooting arguments that parallel those in [10] to directly establish the existence of C^2 solutions for a large class of nonlinearities. Our assumptions on f are essentially the same as those in [10]. We suppose that the restriction of f to real arguments is an odd locally Lipschitz-continuous function with $-\infty < -\sigma^2 \equiv \lim_{s\to 0} \frac{f(s)}{s} \le 0$, and in case $\sigma = 0$ we require that $f(s) < 0$ for small positive s. We assume that the primitive $F(s) \equiv \int_0^s f(t)\,dt$ has exactly one positive zero γ, and that $f(s) > 0$ for $s \in [\gamma,\infty)$. We also assume that $f(s) = \kappa\,|s|^{p-1}\,s + g(s)$, where κ is a positive constant, $p > 1$, and $s^{-p}g(s) \to 0$ as $s \to \infty$. The results in [8] are based on the hypothesis $\sigma = 0$, which results in algebraic decay of solutions

far from the origin, whereas our solutions for $-\sigma^2 < 0$ have exponential decay. We make the assumption $-\infty < -\sigma^2 \equiv \lim_{s\to 0} \frac{f(s)}{s} \le 0$ for simplicity. The conclusion of the Main Theorem which follows remains true if that assumption is replaced by the requirement that $\frac{f(s)}{s}$ is bounded below, with f negative for small positive s.

Because of the rather strong singularity at $r = 0$ in the ordinary differential equation for w, it is not immediately apparent how to formulate a well-posed initial value problem for w with initial conditions given at $r = 0$. To gain insight, we make the change of variable $w(r) = r^m v(r)$ to obtain the equation

$$v'' + \tfrac{2m+1}{r} v' + \tfrac{1}{r^m} f(r^m v) = 0 \tag{1.1}$$

for v, which, by virtue of the condition on f at zero, has well-posed initial value problems obtained by specifying

$$v(0) = d, \text{ and } v'(0) = 0 \tag{1.2}$$

(see Section 2). If v is a C^2 solution of such an initial value problem, it follows that $w(r) = r^m v(r)$ is a C^2 solution of the initial value problem

$$w'' + \tfrac{1}{r} w' - \tfrac{m^2}{r^2} w + f(w) = 0 \tag{1.3}$$

subject to

$$\lim_{r\to 0^+} \tfrac{1}{r^m} w(r) = d \text{ and } \lim_{r\to 0^+} \tfrac{1}{r^{m-1}} w'(r) = md, \tag{1.4}$$

and that the corresponding function $u(r,\theta) = e^{im\theta} w(r)$ is C^2 on R^2.

Note that, although the initial value problem (1.1)-(1.2) is superficially similar to the much-studied radial problem consisting of the differential equation

$$v'' + \tfrac{N-1}{r} v' + f(v) = 0 \tag{1.5}$$

subject to initial conditions (1.2), there is a significant difference between the terms $\frac{1}{r^m} f(r^m v)$ and $f(v)$. Interpreting the differential equations as equations of motion for a point with position $v(r)$ at time r, we note that according to (1.2) the system is released from rest with initial displacement d. The system moving under (1.1) initially experiences the repulsive force $\lim_{r\to 0^+} \frac{-1}{r^m} f(r^m v) = \sigma^2 d$ determined by the behavior of f at the origin, whereas the system moving under (1.5) initially experiences the force $-f(d)$, which is attractive for the values of d that yield solutions that decay at infinity. The character of the problem (1.1)-(1.2) is thus quite different from that of (1.5) with (1.2), and necessitates a separate analysis.

We prove the following main theorem:

MAIN THEOREM: Let the nonlinearity f have the properties specified. Then, for each nonnegative integer n, there is a positive number d and a C^2 solution w to (1.3)-(1.4) such that $\lim_{r\to\infty} w(r) = 0$ and w has exactly n positive zeros.

In section 2, we show that the initial value problem (1.3) - (1.4) has solutions for all $r > 0$. In section 3, we prove that there are values of d for which (1.3) - (1.4) has solutions that are positive for all $r > 0$. In section 4, we show that, similarly, there are values of d for which (1.3) - (1.4) has solutions with any prescribed number of zeros. Section 5 contains the proof of the main theorem. Finally, in section 6, we show that our solutions decay exponentially far from the origin if $-\sigma^2 < 0$.

In the following we denote by α the smallest positive zero of f, and by β the largest positive zero of f. We thus have $0 < \alpha \le \beta < \gamma$. Also, we write $r \to 0$ and $d \to 0$ to mean $r \to 0^+$ and $d \to 0^+$, respectively.

We make repeated use of *Pohozaev's Identity* :

$$r^2(\frac{1}{2}w'^2 + F(w))|_{r_1}^{r_2} = \frac{m^2}{2}w^2|_{r_1}^{r_2} + 2\int_{r_1}^{r_2} sF(w(s))\,ds$$

which results from multiplying (1.3) by r^2w' to obtain $(\frac{1}{2}w'^2)' - (\frac{m^2}{2}w^2)' + r^2(F(w))' = 0$, and then integrating on (r_1, r_2).

2. GLOBAL EXISTENCE OF SOLUTIONS TO THE INITIAL VALUE PROBLEM

We first observe that with the relationship

$$w(r) = r^m v(r), \tag{2.1}$$

which will be used throughout the paper, the initial value problems (1.1)-(1.2) and (1.3)-(1.4) are equivalent. To see this, note first that a simple calculation shows that if v satisfies (1.1)-(1.2) then w satisfies (1.3)-(1.4). Conversely, it is easy to see that if w satisfies (1.3)-(1.4) then v satisfies (1.1) and $v(0) = \lim_{r\to 0} v(r) = d$. To show $v'(0) = 0$ requires a bit more work:

Rewriting (1.3) gives

$$(r^{2m+1}(\frac{w}{r^m})')' = -r^{m+1}f(w).$$

Integrating on $(0, r)$ and noting from (1.4) that

$$\lim_{r\to 0} r^{2m+1}(\frac{w}{r^m})' = \lim_{r\to 0} r^{m+1}w' - mr^m w = 0$$

gives

$$(\frac{w}{r^m})' = \frac{-1}{r^{2m+1}}\int_0^r s^{m+1}f(w)\,ds.$$

Thus,

$$\lim_{r\to 0} v'(r) = \lim_{r\to 0} \frac{-\int_0^r s^{m+1}f(w)\,ds}{r^{2m+1}} = \lim_{r\to 0} -\frac{f(w)}{w}\frac{w}{r^m}\frac{r}{2m+1} = --\sigma^2\ \cdot d\ \cdot 0 = 0.$$

To show the small r existence of solutions to (1.1) - (1.2), note that solutions are fixed points of the mapping G defined by

$$G(v(r)) = d - \int_0^r \frac{1}{s^{2m+1}}\int_0^s t^{m+1}f(t^m v(t))\,dt\,ds.$$

We will now show that G is a contraction mapping for small r. Suppose y and v are continuous on $[0,T]$ with $|y|,|v|<C$. Then

$$|G(y)-G(v)| \le \int_0^T \frac{1}{s^{2m+1}} \int_0^s t^{m+1} |f(t^m y(t)) - f(t^m v(t))| \, dt \, ds$$

$$\le K|y-v| \int_0^T \frac{1}{s^{2m+1}} \int_0^s t^{m+1} t^m \, dt \, ds \le \frac{T^2 K}{4(m+1)} |y-v|$$

where K is the Lipschitz constant for f on $[-C,C]$. Thus, for T small enough we obtain

$$|G(y)-G(v)| < \epsilon |y-v|$$

where $\epsilon < 1$. Thus, by the contraction mapping principle, G has a unique fixed point for T small, and therefore there exists a solution of (1.1) - (1.2) for T small and hence of (1.3) - (1.4) for T small.

To show that the solutions to (1.3) - (1.4) exist for all $r>0$, we show that w and w' remain finite by considering the quantity

$$\tilde{E}(r) = \frac{\frac{1}{2}w'^2(r) + F(w(r)) - F_0}{r^{2m-2}} + \frac{m^2}{2}\frac{w^2(r)}{r^{2m}}$$

where $F_0 = \min F < 0$. Here

$$\tilde{E}(0) = +\infty \text{ and } \tilde{E}(r) \ge 0.$$

A computation shows

$$\tilde{E}' = -\frac{m}{r^{2m-1}}(w' - \frac{mw}{r})^2 - \frac{2(m-1)[F(w(r)) - F_0]}{r^{2m-1}} \le 0.$$

For each solution $w(r)$, $\tilde{E}(r)$ is thus a nonnegative, nonincreasing quantity. Since the term $F(w(r)) - F_0$ is nonnegative, we have that

$$\frac{w'^2}{r^{2m-2}} + \frac{w^2}{r^{2m}} \le 2\tilde{E}(r_0)$$

for all $r \ge r_0$ where r_0 is any positive number. From this and the small r existence of solutions, it follows that the solution $w(r)$ exists for all $r>0$.

3. EXISTENCE OF POSITIVE SOLUTIONS

LEMMA 3.1 : Let w be a nontrivial solution of (1.3) and suppose $w(r_0)=0$. Then there exists a smallest $b>r_0$ such that $|w(b)|=\alpha$. Furthermore, $w \ne 0$ and $w' \ne 0$ on $(r_0, b]$.

PROOF OF LEMMA 3.1 : A nontrivial solution w cannot vanish on any nonempty open interval, by uniqueness of solutions to initial value problems. So, there is an interval $(r_0, r_0+\epsilon)$ on which either $w>0$ and $w'>0$ or $w<0$ and $w'<0$. Without loss of generality, assume $w>0$ and $w'>0$ on $(r_0, r_0+\epsilon)$. There are now two possibilities : either

$$w \text{ has a positive local maximum at some smallest value of } r, \text{ say } r_1, \tag{3.1}$$

or

$$w' \geq 0 \text{ for all } r \geq r_0. \tag{3.2}$$

If (3.1) holds, then at r_1 we have $w'(r_1) = 0$ and $w''(r_1) \leq 0$. Substituting into (1.3) gives

$$-\frac{m^2 w(r_1)}{r_1^2} + f(w(r_1)) \geq 0.$$

Thus $f(w(r_1)) \geq \frac{m^2 w(r_1)}{r_1^2} > 0$ which implies $w(r_1) \geq \alpha$. Therefore, there exists a smallest $b > r_0$, with $b < r_1$, for which $w(b) = \alpha$.

If (3.2) holds then there are two possibilities. Either there exists a smallest $b > r_0$ such that $w(b) = \alpha$ or

$$w'(r) \geq 0, \text{ and } 0 < w(r) < \alpha \text{ for all } r > r_0. \tag{3.3}$$

We claim that (3.3) is impossible. If indeed (3.3) holds, then $f(w) < 0$ for all $r > r_0$ and so we have

$$w'' + \frac{w'}{r} - \frac{m^2 w}{r^2} = -f(w) > 0$$

or equivalently,

$$(r^{2m+1} v')' \geq 0$$

for all $r > r_0$. Integrating twice on (r_0, r) gives :

$$v(r) > v(r_0) + \frac{r_0}{2m} v'(r_0)[1 - (\frac{r_0}{r})^{2m}].$$

If $r_0 = 0$ then $v(r_0) = d > 0$ and $v'(r_0) = 0$, and so we have

$$w(r) \geq d r^m.$$

If $r_0 > 0$ then $v(r_0) = 0$ and $v'(r_0) > 0$, so we have

$$w(r) > \frac{r_0}{2m} v'(r_0)[r^m - \frac{r_0^{2m}}{r^m}].$$

In either case, w grows without bound as r increases, contradicting (3.3).

We have thus established that there exists a smallest $b > r_0$ such that $w(b) = \alpha$. It is clear that $w(r) > 0$ and $w'(r) \geq 0$ for all r in $(r_0, b]$.

To show that $w' \neq 0$ on $(r_0, b]$, suppose on the contrary that $w'(r_2) = 0$ for some r_2 in $(r_0, b]$. Then (1.3) gives $w''(r_2) = \frac{m^2}{r_2^2} - f(w(r_2)) > 0$. Thus, $w(r_2)$ is a local minimum. This contradicts the fact that $w(r) > 0$ and $w'(r) \geq 0$ for all r in $(r_0, b]$.

This completes the proof of Lemma 3.1.

LEMMA 3.2 : Let w be the solution to (1.3)-(1.4) for $d > 0$. Let b_d be the smallest positive value of r for which $w(r) = \alpha$. Then as $d \to 0$, $b_d \to \infty$.

PROOF OF LEMMA 3.2 : Because f is bounded below and $\lim_{w\to 0} \frac{f(w)}{w} = -\sigma^2 < 0$, there exists $M > 0$ such that

$$\frac{f(w)}{w} \geq -M \text{ for all } w.$$

Therefore,

$$w'' + \frac{w'}{r} - \frac{m^2 w}{r^2} = -f(w) \leq Mw \quad \text{for } 0 \leq w \leq \alpha$$

or equivalently

$$v'' + \frac{2m+1}{r} v' \leq Mv \text{ for } 0 \leq r \leq b_d. \tag{3.5}$$

Since $v > 0$ on $[0, b_d]$, dividing by v gives

$$\frac{v''}{v} + \frac{2m+1}{r}\frac{v'}{v} \leq M.$$

Letting

$$y = \log v \tag{3.6}$$

we obtain

$$y'' + \frac{2m+1}{r} y' \leq y'' + y'^2 + \frac{2m+1}{r} y' \leq M.$$

Thus,

$$(r^{2m+1} y')' \leq M r^{2m+1}.$$

Integrating on $(0, r)$ and noting that $\lim_{r\to 0} r^{2m+1} y' = 0$ gives

$$r^{2m+1} y' \leq \frac{M}{2(m+1)} r^{2m+2}.$$

Integrating again on $(0, r)$ gives

$$\log \frac{v}{d} \leq \frac{M}{4(m+1)} r^2.$$

Thus,

$$w \leq d r^m e^{\frac{M}{4(m+1)} r^2} \text{ for } 0 \leq r \leq b_d. \tag{3.7}$$

Evaluating at $r = b_d$ gives

$$\alpha \leq d b_d^m e^{\frac{M}{4(m+1)} b_d^2}.$$

Thus, as $d \to 0$, we have $b_d \to \infty$.

LEMMA 3.3 : Let w and b_d be as in Lemma 3.2. Let a_d be the smallest positive number such that $w(a_d) = \frac{\alpha}{2}$. Then $b_d - a_d \geq K^2 > 0$ where K^2 is a constant that is independent of d if d is small.

PROOF OF LEMMA 3.3 : It follows by evaluating (3.7) at $r = a_d$ that $\lim_{d\to 0} a_d = \infty$. Pohozaev's Identity for $(0, r)$ is

$$\frac{1}{2}r^2 w'^2 + r^2 F(w) = \frac{m^2}{2}w^2 + 2\int_0^r sF(w)\,ds. \tag{3.8}$$

For $a_d \le r \le b_d$ we have $\frac{\alpha}{2} \le w \le \alpha$ and also $F(w) < 0$. Thus,

$$\frac{1}{2}w'^2 + F(w) < \frac{m^2}{2}\frac{\alpha^2}{a_d^2} \text{ for } a_d \le r \le b_d.$$

Now let $C(d) \equiv \frac{m^2}{2}\frac{\alpha^2}{a_d^2}$ and note $C(d) \to 0$ as $d \to 0$. Thus,

$$\frac{w'^2}{C(d) - F(w)} < 2 \text{ for } a_d \le r \le b_d.$$

We saw in the proof of Lemma 3.1 that $w' \ge 0$ for $0 \le r \le b_d$. So, taking square roots of the above and integrating on (a_d, b_d) gives :

$$\int_{\frac{\alpha}{2}}^{\alpha} \frac{ds}{\sqrt{C(d) - F(s)}} = \int_{a_d}^{b_d} \frac{w'}{\sqrt{C(d) - F(w)}}\,dr < \sqrt{2}(b_d - a_d).$$

Now as $d \to 0$ the left hand side of the above approaches the constant

$$\int_{\frac{\alpha}{2}}^{\alpha} \frac{ds}{\sqrt{-F(s)}} > 0.$$

Thus for d small enough we have

$$b_d - a_d \ge K^2 > 0.$$

This completes the proof of Lemma 3.3.

LEMMA 3.4 : Let w be the solution of (1.3)-(1.4) for $d > 0$. For d chosen small enough, we have $0 < w(r) < \gamma$ for all $r > 0$.

PROOF OF LEMMA 3.4 : Recall that $w(r) > 0$ and $w'(r) > 0$ for r small. We first claim that if $w(r) < \gamma$ for r in $(0, c)$ then $w(r) \ne 0$ for r in $(0, c)$.

To establish this fact, suppose that $w(r) = 0$ for some r in $(0, c)$ and let $z_d \in (0, c)$ be the smallest such value of r. Pohozaev's Identity on $(0, z_d)$ is

$$\frac{1}{2}z_d^2 w'^2(z_d) = 2\int_0^{z_d} rF(w(r))\,dr.$$

Since $0 < w(r) < \gamma$ for r in $(0, z_d)$, $F(w(r)) < 0$ for r in $(0, z_d)$. Thus the right hand side is negative, whereas the left hand side is nonnegative. Thus, there is no zero of $w(r)$ in the interval $(0, c)$ if $w(r) < \gamma$ on $(0, c)$.

We next claim that for sufficiently small d, $w(r) < \gamma$ for all $r > 0$. To establish this fact, suppose there is a smallest value c_d of r such that $w(c_d) = \gamma$. Then, as we just established, $0 < w < \gamma$ on $(0, c_d)$.

Pohozaev's Identity on $(0, c_d)$ is :

$$0 \leq \frac{1}{2}c_d^2 w'^2(c_d) = \frac{m^2}{2}\gamma^2 + 2\int_0^{c_d} rF(w)\,dr. \tag{3.9}$$

We will now show that the right hand side is negative for sufficiently small d, whereas the left hand side is nonnegative, resulting in a contradiction to the supposition $w(c_d) = \gamma$. We have the following inequalities

$$0 < a_d < b_d < c_d$$

and $F(w) \leq 0$ on $(0, c_d)$. Further, for $a_d \leq r \leq b_d$ we have $\frac{\alpha}{2} \leq w \leq \alpha$ and thus $F(w) \leq F(\frac{\alpha}{2}) < 0$ since F is decreasing on $[\frac{\alpha}{2}, \alpha]$. Thus,

$$\int_0^{c_d} rF(w)\,dr \leq \int_{a_d}^{b_d} rF(w)\,dr \leq \frac{1}{2}F(\frac{\alpha}{2})(b_d^2 - a_d^2). \tag{3.10}$$

Now from Lemmas 3.2 and 3.3 we have

$$b_d^2 - a_d^2 = (b_d - a_d)(b_d + a_d) \geq K^2(b_d + a_d) \geq K^2 b_d \to \infty \text{ as } d \to 0.$$

Since $F(\frac{\alpha}{2}) < 0$ we have

$$\int_0^{c_d} rF(w)\,dr \to -\infty \text{ as } d \to 0.$$

Hence, the right hand side of (3.9) is negative for d small enough and this gives the desired contradiction.

4. SOLUTIONS WITH PRESCRIBED NUMBER OF ZEROS

In this section, we show that there are solutions of (1.3) - (1.4) with an arbitrarily large number of zeros. To do this, we study the behavior of solutions as d grows large. In this section, given $\lambda > 0$, let $z_\lambda(r)$ be the solution of (1.3) - (1.4) with $d \equiv \lambda^{\frac{2}{p-1}+m}$. Then define

$$y_\lambda(r) = \lambda^{-\frac{2}{p-1}} r^{-m} z_\lambda(\frac{r}{\lambda}). \tag{4.1}$$

Then y_λ satisfies

$$y_\lambda'' + \frac{2m+1}{r}y_\lambda' + \lambda^{-\frac{2}{p-1}-2} r^{-m} f(\lambda^{\frac{2}{p-1}} r^m y_\lambda) = 0 \tag{4.2}$$

and

$$y_\lambda(0) = 1, \; y_\lambda'(0) = 0. \tag{4.3}$$

Now we make use of the hypothesis

$$f(w) = \kappa|w|^{p-1}w + g(w) \tag{4.4}$$

where $\lim_{|w|\to\infty} \frac{g(|w|)}{|w|^p} = 0$.

LEMMA 4.1 : As $\lambda \to \infty$, $y_\lambda \to y$ uniformly on compact subsets of $[0,\infty)$, where y is the solution of

$$y'' + \frac{2m+1}{r} y' + \kappa r^{m(p-1)} |y|^{p-1} y = 0 \tag{4.5}$$

and

$$y(0) = 1, \ y'(0) = 0. \tag{4.6}$$

PROOF OF LEMMA 4.1 : As in Section 2, we can show that for each $\lambda > 0$, (4.2) - (4.3) has a solution for all $r > 0$. We can also define a decreasing energy by

$$E_\lambda(r) = \frac{1}{2}(r y_\lambda' + m y_\lambda)^2 + \frac{m^2}{2} y_\lambda^2 + \lambda^{-\frac{4}{p-1}-2} \frac{F(\lambda^{\frac{2}{p-1}} r^m y_\lambda)}{r^{2m-2}} + 2(m-1)\lambda^{-\frac{4}{p-1}-2} \int_0^r \frac{F(\lambda^{\frac{2}{p-1}} s^m y_\lambda)}{s^{2m-1}}\, ds \tag{4.7}$$

Here,

$$E_\lambda' = -r y_\lambda'^2 \le 0, \text{ and } E_\lambda(0) = \lim_{r\to 0} E_\lambda(r) = m^2.$$

Now we will show that the integral term in the above energy is bounded below as $\lambda \to \infty$. This will allow us to bound the other terms from above and then appeal to the Arzela-Ascoli Theorem to obtain a convergent subsequence of the $\{y_\lambda\}$.

To this end recall the following inequalities which were established in Lemma 3.1 and equation (3.6):

$$d r^m \le w(r) \le d r^m e^{\frac{M}{4(m+1)} r^2} \text{ for } 0 \le r \le b_d \tag{4.8}$$

where w is a solution of (1.3)-(1.4) with $d > 0$, and b_d is the first value of r such that $w(b_d) = \alpha$. In terms of y_λ, this becomes

$$1 \le y_\lambda(r) \le e^{\frac{M}{4(m+1)} \frac{r^2}{\lambda^2}} \text{ for } 0 \le r \le \lambda b_d. \tag{4.9}$$

We split the integration interval in (4.7) into the subintervals $(0, \lambda b_d)$ and $(\lambda b_d, r)$. For the integral over $(0, \lambda b_d)$, we use (4.9) and the fact that

$$\frac{F(s)}{s^2} \ge -C_1$$

for some $C_1 > 0$ and we obtain

$$\lambda^{-\frac{4}{p-1}-2} \int_0^{\lambda b_d} \frac{F(\lambda^{\frac{2}{p-1}} s^m y_\lambda)}{s^{2m-1}}\, ds \ge -C_1 \lambda^{-\frac{4}{p-1}-2} \int_0^{\lambda b_d} \frac{\lambda^{\frac{4}{p-1}} s^{2m} y_\lambda^2}{s^{2m-1}}\, ds$$

$$\ge -\frac{C_1}{\lambda^2} \int_0^{\lambda b_d} s e^{\frac{M}{2(m+1)} \frac{s^2}{\lambda^2}}\, ds = -\frac{C_1(m+1)}{M} [e^{\frac{M}{2(m+1)} b_d^2} - 1].$$

Now substituting $r = b_d$ into (4.8) and recalling that $d = \lambda^{\frac{2}{p-1}+m}$ we obtain

$$\lambda^{\frac{2}{p-1}+m} b_d^m \leq w(b_d) = \alpha.$$

Hence

$$b_d \to 0 \text{ as } \lambda \to \infty.$$

Thus, we see that

$$-\frac{C_1(m+1)}{M}[e^{\frac{M}{2(m+1)}b_d^2} - 1] \to 0 \text{ as } \lambda \to \infty.$$

To estimate the integral over $(\lambda b_d, r)$ we recall that

$$F(s) \geq -C_2 \text{ for some } C_2 > 0,$$

so that

$$\lambda^{-\frac{4}{p-1}-2} \int_{\lambda b_d}^{r} \frac{F(\lambda^{\frac{2}{p-1}} s^m y_\lambda)}{s^{2m-1}}\, ds \geq -C_2 \lambda^{-\frac{4}{p-1}-2} \int_{\lambda b_d}^{r} \frac{1}{s^{2m-1}}\, ds \geq -\frac{C_2 \lambda^{-\frac{4}{p-1}-2}}{2m-2} \frac{1}{(\lambda b_d)^{2m-2}}.$$

Returning once more to (4.8) and letting $r = b_d$ we obtain

$$\frac{\alpha}{\lambda^{\frac{2}{p-1}} e^{\frac{M}{4(m+1)}b_d^2}} \leq (\lambda b_d)^m.$$

Thus,

$$-\lambda^{-\frac{4}{p-1}-2} \frac{1}{(\lambda b_d)^{2m-2}} \geq -\frac{(\lambda b_d)^2}{\lambda^{\frac{4}{p-1}+2}} \frac{\lambda^{\frac{4}{p-1}} e^{\frac{M}{2(m+1)}b_d^2}}{\alpha^2} = -\frac{b_d^2 e^{\frac{M}{2(m+1)}b_d^2}}{\alpha^2}.$$

As we saw above, $b_d \to 0$ as $\lambda \to \infty$. Thus, this second integral is also bounded below by a constant which goes to zero as $\lambda \to \infty$.

Thus, we see that

$$m^2 \geq E_\lambda(r) \geq y_\lambda^2[\frac{m^2}{2} - C_1 \frac{r^2}{\lambda^2}] + C_3(\lambda) \quad \text{where } C_3(\lambda) \to 0 \text{ as } \lambda \to \infty.$$

Thus, for r in any compact set, if λ is chosen large enough we have

$$y_\lambda^2 \leq M^2$$

where M is independent of λ.

Next we will show that $y_\lambda'^2 \leq C$. Multiplying (4.2) by r^{2m+1} and integrating gives:

$$-y_\lambda' = \frac{\lambda^{\frac{-2}{p-1}-2}}{r^{2m+1}} \int_0^r s^{m+1} f(\lambda^{\frac{2}{p-1}} s^m y_\lambda)\, ds$$

Since

$$|f(w)| \leq C|w|^p + D|w|,$$

for some positive constants C and D, we have

$$|f(\lambda^{\frac{2}{p-1}} s^m y_\lambda)| \leq C\lambda^{\frac{2p}{p-1}} s^{pm} M^p + D\lambda^{\frac{2}{p-1}} s^m M$$

for sufficiently large λ. Thus,

$$|y_\lambda'| \leq \frac{A}{r^{2m+1}} \int_0^r s^{pm+m+1}\, ds + \frac{B}{\lambda^2 r^{2m+1}} \int_0^r s^{2m+1}\, ds \leq \frac{A}{pm+m+2} r^{(p-1)m+1} + \frac{B}{2m+2} \frac{r}{\lambda^2}.$$

Thus, on compact sets we have

$$y_\lambda^2, y_\lambda'^2 \leq C.$$

So, by the Arzela-Ascoli Theorem, there exists a subsequence (again labeled by λ) such that the $\{y_\lambda\}$ converge uniformly as $\lambda \to \infty$ to some continuous function y. It remains to show that y satisfies (4.5) - (4.6).

Since y_λ is a solution of (4.2) - (4.3) we have

$$-r^{2m+1} y_\lambda' = \int_0^r \lambda^{-\frac{2}{p-1}-2} s^{m+1} f(\lambda^{\frac{2}{p-1}} s^m y_\lambda)\, ds.$$

Since $y_\lambda \to y$ uniformly and f is of the form (4.4), we see that the right hand side of the above converges to

$$j(r) = \int_0^r \kappa s^{1+m+pm} |y|^{p-1} y\, ds.$$

Thus, the sequence $\{y_\lambda'\}$ converges pointwise to the function $\frac{-j(r)}{r^{2m+1}}$. Since y is continuous, so is j. Further, since the $\{y_\lambda'\}$ are uniformly bounded on compact sets, we can apply the dominated convergence theorem to

$$y_\lambda(r) = 1 + \int_0^r y_\lambda'(s)\, ds$$

and deduce that

$$y(r) = 1 - \int_0^r \frac{j(s)}{s^{2m+1}}\, ds.$$

Therefore, $y'(r) = \frac{-j(r)}{r^{2m+1}}$. Thus,

$$y(0) = 1,$$

$$y'(0) = \lim_{r \to 0} y'(r) = \lim_{r \to 0} \frac{-\int_0^r \kappa s^{1+m+pm} |y|^{p-1} y\, ds}{r^{2m+1}} = \lim_{r \to 0} \frac{-\kappa r^{1+m+pm} |y|^{p-1} y}{(2m+1) r^{2m}} = 0,$$

and

$$r^{2m+1} y' = -\int_0^r \kappa s^{1+m+mp} |y|^{p-1} y\, ds.$$

This is equivalent to (4.5) - (4.6).

LEMMA 4.2 : Let y be a solution of (4.5) - (4.6). Then y has at least one zero.

PROOF OF LEMMA 4.2 : It will be somewhat simpler to show that $z = r^m y$ has at least one zero for $r > 0$. We use an argument based on that of Proposition 3.9 of [5]. The function z satisfies :

$$z'' + \frac{1}{r}z' - \frac{m^2}{r^2}z + \kappa|z|^{p-1}z = 0$$

and

$$\lim_{r\to 0}\frac{z}{r^m} = 1, \ \lim_{r\to 0}\frac{z'}{r^{m-1}} = m.$$

Multiplying by $r^2 z'$ and integrating on $(0, r)$ gives Pohozaev's Identity :

$$\frac{1}{2}r^2 z'^2 - \frac{m^2}{2}z^2 + \frac{\kappa}{p+1}r^2|z|^{p+1} = \frac{2\kappa}{p+1}\int_0^r s|z|^{p+1}\,ds. \tag{4.10}$$

Now we assume that $y > 0$ for $r > 0$ (thus $z > 0$ for $r > 0$) and we will show that $z \to 0$, $r^2|z|^{p+1} \to 0$ as $r \to \infty$ and

$$\int_0^\infty s|z|^{p+1}\,ds < \infty.$$

Using (4.10), this will show that $r^2 z'^2$ has a nonzero limit and this will lead to a contradiction.

Assuming now that $y > 0$, multiplying (4.5) by r^{2m+1} and integrating on $(0, r)$ gives:

$$-r^{2m+1}y' = \int_0^r \kappa s^{m+mp+1}y^p\,ds. \tag{4.11}$$

Thus, (4.13) implies $y' < 0$. Hence, y is decreasing. So,

$$-r^{2m+1}y' = \int_0^r \kappa s^{m+mp+1}y^p\,ds \geq y^p\int_0^r \kappa s^{m+mp+1}\,ds = \frac{\kappa y^p r^{m+mp+2}}{m+mp+2}.$$

Dividing by $r^{2m+1}y^p$ and integrating on $(0, r)$ gives

$$\frac{y^{-p+1}-1}{p-1} \geq \frac{\kappa r^{mp-m+2}}{(m+mp+2)(mp-m+2)}.$$

Hence,

$$y \leq \frac{C_{m,p}}{r^{m+\frac{2}{p-1}}} \text{ where } C_{m,p} = [\frac{(m+mp+2)(mp-m+2)}{c(p-1)}]^{\frac{1}{p-1}}.$$

Therefore, $z \leq C_{m,p}r^{-\frac{2}{p-1}}$ so,

$$\lim_{r\to\infty} z = 0. \tag{4.12}$$

Also, $r^2 z^{p+1} \leq C_{m,p}^{p+1}r^{-\frac{4}{p-1}}$ so,

$$\lim_{r\to\infty} r^2 z^{p+1} = 0. \tag{4.13}$$

Further,

$$\int_1^\infty rz^{p+1} \leq \int_1^\infty \frac{C_{m,p}^{p+1}}{r^{1+\frac{4}{p-1}}} = \frac{C_{m,p}^{p+1}(p-1)}{4}. \tag{4.14}$$

Now using (4.12) - (4.14), we can take limits in (4.10) and obtain

$$\lim_{r\to\infty} \frac{1}{2} r^2 z'^2 = \frac{2\kappa}{p+1} \int_0^\infty r|z|^{p+1} = L < \infty.$$

Clearly, $L \geq 0$. If $L = 0$ then $\int_0^\infty r|z|^{p+1} = 0$ which implies $z \equiv 0$. But this contradicts the fact that $z > 0$. On the other hand, if $L > 0$, then

$$r^2 z'^2 \geq L \text{ for large } r.$$

Thus, $|z'| > 0$ for large r, and since $\lim_{r\to\infty} z = 0$ and $z > 0$ we must then have $z' < 0$ for large r. Therefore, for some r_0 we have

$$-z' \geq \frac{\sqrt{L}}{r} \text{ for } r \geq r_0.$$

Hence,

$$z(r_0) - z(r) \geq \sqrt{L} \log \frac{r}{r_0} \text{ for } r \geq r_0.$$

This implies $z(r) \to -\infty$ as $r \to \infty$. Therefore, we obtain a contradiction to the assumption that $z > 0$ for all $r > 0$, and thus z, and hence y, has at least one positive zero. This completes the proof of Lemma 4.2.

LEMMA 4.3 : Let y be a solution of (4.5) - (4.6). Then y has infinitely many zeros.

PROOF OF LEMMA 4.3 : From Lemma 4.2, we know there is a smallest $r_1 > 0$ such that $z(r_1) = 0$ where $z = r^m y$. By virtue of (4.10) we know that $z'(r_1) = -A < 0$. Recall that

$$\tilde{E} = \frac{1}{2} \frac{z'^2}{r^{2m-2}} + \frac{\kappa}{p+1} \frac{|z|^{p+1}}{r^{2m-2}} + \frac{m^2}{2} \frac{z^2}{r^{2m}}$$

is a decreasing energy for (4.5). Thus,

$$\tilde{E}(r) \leq \tilde{E}(r_1) = \frac{1}{2} \frac{A^2}{r_1^{2m-2}}$$

for $r \geq r_1$. In particular, for $r \geq r_1$ we have,

$$\frac{|z|^{p+1}}{r^{2m-2}} \leq \frac{p+1}{2\kappa} \frac{A^2}{r_1^{2m-2}}.$$

Thus y satisfies (4.5) and :

$$y(r_1) = 0, \; y'(r_1) = -\frac{A}{r_1^m}.$$

The bound on z for $r \geq r_1$ yields a corresponding estimate for y :

$$|y|^{p+1} \leq \frac{C}{r^{m(p-1)+2}}.$$

Thus, $y \to 0$ as $r \to \infty$, and so it follows that y must have a local minimum, y_1, at $r = t_1 > r_1$. So y satisfies the initial value problem consisting of (4.5) subject to

$$y(t_1) = y_1, \; y'(t_1) = 0.$$

We may now use the same argument as in Lemma 4.2, replacing $r = 0$ with $r = t_1$ to show that y has a second zero at $r = r_2 > r_1$. Proceeding inductively, we can show that y has infinitely many zeros.

LEMMA 4.4 : Denote by $w(r, d)$ the solution to (1.3) - (1.4). Let d_0 be a value for which $w(r, d_0)$ has exactly k zeros and $w(r, d_0) \to 0$ as $r \to \infty$. If $|d - d_0|$ is sufficiently small, then $w(r, d)$ has at most $k + 1$ zeros on $[0, \infty)$.

PROOF OF LEMMA 4.4 : The proof is similar to those for Lemmas 3.1 - 3.5. We wish to show that for d near d_0, $w(\cdot, d)$ has at most $(k + 1)$ zeros in $[0, \infty)$. So we suppose there is a sequence of values d_j converging to d_0 such that $w(\cdot, d_j)$ has at least $(k+1)$ zeros in $[0, \infty)$. (If there is no such sequence, then the lemma is proven.) We write $w_j(r) \equiv w(r, d_j)$ and we denote by r_j the $(k + 1)$st zero of w_j, counting from the smallest.

Because of the continuous dependence of the solution on initial conditions, we know that $w_j \to w_0 \equiv w(\cdot, d_0)$ and $w'_j \to w'_0$ uniformly on compact sets as $j \to \infty$. In particular, let $[0, L)$ be any bounded interval containing the k zeros of w_0. Then, for sufficiently large j, the function w_j has exactly k zeros in $[0, L)$. Thus, $r_j \to \infty$ as $j \to \infty$.

Let b_j be the smallest number greater than r_j such that $|w_j(b_j)| = \alpha$. The existence of b_j is guaranteed by Lemma 3.1. Let a_j be the smallest number greater than r_j such that $|w_j(a_j)| = \frac{\alpha}{2}$.

Let q_j be the largest number less than r_j such that $|w_j(q_j)| = \gamma$. That there is such a number q_j can be seen as follows. Let p_j be the location of a local extremum between the kth and $(k + 1)$st zeros of w_j. Evaluating Pohozaev's Identity between p_j and r_j gives

$$0 < \frac{1}{2} r_j^2 w_j'^2(r_j) + \frac{m^2}{2} w_j^2(p_j) = p_j^2 F(w(p_j)) + \int_{p_j}^{r_j} sF(w(s))\, ds.$$

It follows that $F(w(r)) > 0$ for some r in (p_j, r_j), and hence $|w(r)| > \gamma$ for some r in (p_j, r_j). Thus, there is a largest number q_j less than r_j such that $|w_j(q_j)| = \gamma$.

As in Lemma 3.3, we may now verify the following.

CLAIM : $b_j - a_j \geq K^2 > 0$, where K is a constant independent of j for j sufficiently large.

PROOF OF CLAIM : Evaluating Pohozaev's Identity between q_j and r gives

$$r^2 [\frac{1}{2} w_j'^2(r) + F(w_j(r))] = \frac{m^2}{2} (w_j^2(r) - \gamma^2) + \frac{1}{2} q_j^2 w_j'^2(q_j) + 2 \int_{q_j}^{r} sF(w_j(s))\, ds.$$

Using the facts that $|w_j(r)| \leq \gamma$ and $F(w_j(r)) \leq 0$ for $q_j \leq r \leq b_j$, we obtain

$$\frac{1}{2} w_j'^2(r) + F(w_j(r)) \leq \frac{1}{2} w_j'^2(q_j)$$

Now, $|w_j(q_j)| = \gamma$, and w_j tends to w_0 uniformly on compact sets as $j \to \infty$. Since $w_0(r) \to 0$ as $r \to \infty$, it follows that q_j has the finite limit q_0 as $j \to \infty$, where $|w_0(q_0)| = \gamma$. Hence, $w'_j(q_j) \to w'_0(q_0)$ as

$j \to \infty$, so that

$$\frac{1}{2}w_j'^2(r) + F(w_j(r)) \le D \tag{4.15}$$

for $q_j \le r \le b_j$, where $D > 0$ is a constant that is independent of j for sufficiently large j.

Now Lemma 3.1 shows that $w_j'(r) \neq 0$ for r in $(r_j, b_j]$ so from (4.15) we have

$$0 < \int_{\frac{\alpha}{2}}^{\alpha} \frac{dy}{\sqrt{D - F(y)}} = \int_{a_j}^{b_j} \frac{|w_j'(r)|\, dr}{\sqrt{D - F(w_j(r))}} \le \int_{a_j}^{b_j} \sqrt{2}\, dr = \sqrt{2}(b_j - a_j)$$

for sufficiently large j. This proves the claim, with $K^2 \equiv \frac{1}{2\sqrt{2}} \int_{\frac{\alpha}{2}}^{\alpha} \frac{dy}{\sqrt{D-F(y)}}$

As in Lemma 3.4, we may verify the following.

CLAIM : For sufficiently large j, $|w_j(r)| < \gamma$ for all $r > r_j$.

PROOF OF CLAIM : Suppose on the contrary that there is a smallest $c_j > r_j$ such that $|w_j(c_j)| = \gamma$. Evaluating Pohozaev's Identity between q_j and c_j gives

$$\frac{1}{2}c_j^2 w_j'^2(c_j) = \frac{1}{2}q_j^2 w_j'^2(q_j) + 2\int_{q_j}^{c_j} sF(w_j(s))\, ds. \tag{4.16}$$

Since $F(w(s)) \le 0$ on (c_j, q_j), and F is decreasing on $[\frac{\alpha}{2}, \alpha]$, as in Lemma 3.4 we have

$$\int_{q_j}^{c_j} sF(w_j(s))\, ds \le \int_{a_j}^{b_j} sF(w_j(s))\, ds \le \frac{1}{2}F(\frac{\alpha}{2})(b_j^2 - a_j^2) \le \frac{K^2}{2}F(\frac{\alpha}{2})(b_j + a_j).$$

Now $q_j^2 w_j'^2(q_j) \to q_0^2 w_0'^2(q_0)$ and $a_j + b_j \to \infty$ as $j \to \infty$. Thus, the right-hand side of (4.16) tends to $-\infty$ as $j \to \infty$, whereas the left-hand side is positive. This contradicts the assumption that there is $c_j > r_j$ with $|w_j(c_j)| = \gamma$, and proves the claim.

Finally, to complete the proof of Lemma 4.4, suppose that w_j has another zero $t_j > r_j$. Then there is a local extremum for w_j at a location s_j with $r_j < s_j < t_j$. Evaluating Pohozaev's Identity between s_j and t_j gives

$$\frac{1}{2}t_j^2 w_j'^2(t_j) + \frac{m^2}{2}w_j^2(s_j) = s_j^2 F(w_j(s_j)) + 2\int_{s_j}^{t_j} sF(w_j(s))\, ds.$$

This implies that $F(w_j(r)) > 0$ for some r between s_j and t_j. But for sufficiently large j, $|w_j(r)| < \gamma$ for all $r > r_j$, hence $F(w_j(r)) < 0$ for r between s_j and t_j. This contradiction shows that for j sufficiently large, there is no zero of w_j larger than r_j. This completes the proof of Lemma 4.4

5. PROOF OF THE MAIN THEOREM

PROOF OF MAIN THEOREM : We define

$$A_0 = \{d > 0 | w(r, d) \text{ has no positive zeros}\}. \tag{5.1}$$

Lemma 3.4 shows that the set A_0 is nonempty. Also, Lemmas 4.1 - 4.3 imply that the set A_0 is bounded above. Let $d_0 = \sup A_0$. We will show that

$$w(r, d_0) > 0, \text{ and } w(r, d_0) \to 0 \text{ as } r \to \infty.$$

First, if $w(r, d_0)$ has a zero at some finite r, then continuity of $w(r, d)$ on d implies that $w(r, d)$ has a zero for d slightly smaller than d_0. This contradicts the definition of d_0. Thus,

$$w(r, d_0) > 0 \text{ for all } r > 0. \tag{5.3}$$

Next, either

$$w \text{ is monotone for large } r$$

or

$$w(r, d_0) \text{ has positive local minima at arbitrarily large values of } r.$$

We will show this second case is impossible.

CLAIM : If w has positive local minima at M_k where $M_k \to \infty$, then $\limsup_{k\to\infty} w(M_k) \le \beta$.

PROOF OF CLAIM : At a minimum M_k of w, we have $w'(M_k) = 0$ and $w''(M_k) \ge 0$. Substituting into (1.3) gives

$$-\frac{m^2}{M_k^2} w(M_k) + f(w(M_k)) \le 0.$$

Thus, since $w(M_k) > 0$, we obtain

$$\frac{f(w(M_k))}{w(M_k)} \le \frac{m^2}{M_k^2}.$$

Thus, since $M_k \to \infty$ as $k \to \infty$ we have

$$\limsup_{k\to\infty} \frac{f(w(M_k))}{w(M_k)} \le 0.$$

This implies

$$\limsup_{k\to\infty} w(M_k) \le \beta.$$

This completes the proof of CLAIM.

Now we will show that it is impossible for $w(r, d_0)$ to have positive local minima at arbitrarily large values of r. From the above CLAIM, for sufficiently large k we have

$$w(M_k, d_0) \le \frac{3}{4}\beta + \frac{1}{4}\gamma.$$

Now, we choose d slightly larger than d_0 so that $w(r, d)$ has a zero at z. By virtue of continuous dependence on initial values, for d close enough to d_0, $w(r, d)$ will also have positive local minima at N_k for $k = 1, \cdots, K$,

where N_K is the largest value of r less than z at which w has a minimum. Evaluating Pohozaev's Identity between 0 and z gives

$$0 \leq \frac{1}{2}z^2 w'^2(z) = 2\int_0^z rF(w)\,dr. \tag{5.6}$$

Also, evaluating between 0 and N_K gives

$$N_K^2 F(w(N_K)) = \frac{m^2}{2}w^2(N_K) + 2\int_0^{N_K} rF(w)\,dr. \tag{5.7}$$

We may furthermore choose d sufficiently close to d_0 so that

$$w(N_K, d) \leq \frac{1}{2}\beta + \frac{1}{2}\gamma < \gamma. \tag{5.8}$$

Thus,

$$F(w(N_K)) \leq 0.$$

So we see from (5.7) that

$$2\int_0^{N_K} rF(w)\,dr \leq 0. \tag{5.9}$$

Combining (5.6) and (5.9) we have

$$2\int_{N_K}^z rF(w)\,dr \geq 0. \tag{5.10}$$

Thus, there exists a c with

$$N_K < c < z \text{ such that } w(c) = \gamma.$$

Further, c may be chosen so that $w' \geq 0$ on $[N_K, c]$ because N_K is the largest value of r at which w has a minimum before the zero, z, of w. Pohozaev's Identity evaluated between N_K and c gives

$$0 \leq \frac{1}{2}c^2 w'^2(c) = N_K^2 F(w(N_K)) + \frac{m^2}{2}(\gamma^2 - w^2(N_K)) + 2\int_{N_K}^c rF(w)\,dr \tag{5.12}$$

$$\leq \frac{m^2}{2}(\gamma^2 - w^2(N_K)) + 2\int_{N_K}^c rF(w)\,dr.$$

We will now show that the right hand side of (5.12) is negative for d close enough to d_0 and thus obtain a contradiction to the assumption that $w(r, d_0)$ has local minima at arbitrarily large values of r.

We will estimate the integral on the right. Let a, b be such that $N_K < a < b < c$ and $w(a) = \frac{1}{2}\beta + \frac{1}{2}\gamma,\ w(b) = \frac{1}{4}\beta + \frac{3}{4}\gamma$. Then since $w' \geq 0$, and $F(w) \leq 0$ on $[a, b]$, and $f(w) = F'(w) \geq 0$ on $[\frac{\beta+\gamma}{2}, \frac{\beta+3\gamma}{4}]$ we obtain :

$$2\int_{N_K}^c rF(w)\,dr \leq 2\int_a^b rF(w)\,dr \leq F(\frac{\beta + 3\gamma}{4})(b^2 - a^2).$$

Thus, from (5.12) we see that

$$0 \leq \frac{m^2}{2}(\gamma^2 - w^2(N_K)) + F(\frac{\beta + 3\gamma}{4})(b^2 - a^2). \tag{5.13}$$

We will next show that $b - a \geq \epsilon > 0$ where ϵ is independent of d. To show this, we note that for $N_K \leq r \leq c$ we have

$$\int_0^r 2sF(w)\,ds = \int_0^{N_K} 2sF(w)\,ds + \int_{N_K}^r 2sF(w)\,ds.$$

From (5.9) we see that the first integral on the right is negative. Also, the second integral is negative because $0 < w \leq \gamma$ on $[N_K, c]$ and so $F(w) \leq 0$. Thus, Pohozaev's Identity between 0 and r gives

$$\frac{1}{2}w'^2 + F(w) \leq \frac{m^2}{2}\frac{w^2}{r^2} \text{ for } N_K \leq r \leq c.$$

Choosing d close enough to d_0 we can always ensure that $N_K \geq 1$. Thus,

$$\frac{1}{2}w'^2 + F(w) \leq C \equiv \frac{m^2}{2}\gamma^2 \text{ for } N_K \leq r \leq c.$$

Since $w' \geq 0$ on $[N_K, c]$, we obtain

$$\frac{w'}{\sqrt{C - F(w)}} \leq \sqrt{2} \text{ for } N_K \leq r \leq c.$$

Now since $[a, b] \subset [N_K, c]$, we can integrate on $[a, b]$ and get

$$\int_{\frac{\beta+\gamma}{2}}^{\frac{\beta+3\gamma}{4}} \frac{ds}{\sqrt{C - F(s)}} = \int_a^b \frac{w'}{\sqrt{C - F(w)}}\,dr \leq \int_a^b \sqrt{2}\,dr = \sqrt{2}(b - a).$$

We have thus established that $b - a \geq \frac{1}{\sqrt{2}} \int_{\frac{\beta+\gamma}{2}}^{\frac{\beta+3\gamma}{4}} \frac{ds}{\sqrt{C-F(s)}} = \epsilon > 0$, as claimed.

Now, since $b^2 - a^2 = (b - a)(b + a)$ and $b + a > 2N_K \to \infty$ as $d \to d_0$ we obtain that $b^2 - a^2 \to \infty$ as $d \to d_0$. Therefore, the right hand side of (5.13) approaches $-\infty$ as $d \to d_0$. This is the required contradiction and hence $w(r, d_0)$ must be monotone for large values of r.

Next, an argument similar to that in the proof of Lemma 3.2 shows that the monotonicity of $w(r, d_0)$ for large r implies that $w(r, d_0)$ is bounded in r. We thus have

$$\lim_{r\to\infty} w(r, d_0) = \zeta \geq 0. \tag{5.14}$$

Taking limits in

$$\frac{1}{2}w'^2 + F(w) = \frac{m^2}{2}\frac{w^2}{r^2} + \frac{2}{r^2}\int_0^r sF(w)\,ds$$

we obtain

$$\lim_{r\to\infty} \frac{1}{2}w'^2 + F(\zeta) = F(\zeta).$$

(When taking the limit of the right-hand side we use the following fact : if g is continuous and $\lim_{r\to\infty} g(r) = g_0$, then $\lim_{r\to\infty} \frac{2}{r^2}\int_0^r sg(s)\,ds = g_0$. Proof : $|\frac{2}{r^2}\int_0^r s[g(s) - g_0]\,ds| \leq \frac{2}{r^2}\int_0^R s|g(s) - g_0|\,ds + \frac{2}{r^2}\int_R^r s|g(s) - g_0|\,ds \leq \frac{2MR^2}{r^2} + \frac{2}{r^2}\int_R^r s\epsilon\,ds = \frac{2MR^2}{r^2} + \epsilon$.) Thus,

$$\lim_{r\to\infty} w'(r, d_0) = 0. \tag{5.15}$$

Also, from (1.3) it follows that

$$\lim_{r\to\infty} w'' = -f(\zeta).$$

So, we must have

$$f(\zeta) = 0. \tag{5.16}$$

We finally need to show that $\zeta = 0$. If not, that is if $\zeta > 0$, then since $w(r,d) \to w(r,d_0)$ uniformly on compact sets, given $\epsilon > 0$ we can choose a large value, q, and $d > d_0$ sufficiently close to d_0 such that

$$\zeta - \epsilon \leq w(q,d) \leq \zeta + \epsilon.$$

Now, we know $w(r,d)$ has a zero $z > q$. Evaluating Pohozaev's Identity between q and z gives

$$0 \leq \frac{1}{2}z^2 w'^2(z) = q^2[\frac{1}{2}w'^2(q) - \frac{m^2}{2}\frac{w^2(q)}{q^2} + F(w(q))] + 2\int_q^z rF(w)\,dr.$$

For q sufficiently large, and for d close enough to d_0 but slightly larger than d_0, the term in brackets is close to $F(\zeta) < 0$ because $w(r,d) \to w(r,d_0)$ uniformly on compact sets and $w(r,d_0) \to \zeta$ and $w'(r,d_0) \to 0$ as $r \to \infty$. Thus,

$$\int_q^z rF(w) \geq 0.$$

So, there is an s with $q < s < z$ such that $w(s) = \gamma$ and $\beta < w(r) < \gamma$ on (q,s). Evaluating Pohozaev's Identity between q and s, and noting that $F(w) \leq 0$ for $r \in [q,s]$, we obtain :

$$0 \leq \frac{1}{2}s^2 w'^2(s) \leq q^2[\frac{1}{2}w'^2(q) + \frac{m^2}{2}\frac{\gamma^2}{q^2} - \frac{m^2}{2}\frac{w^2(q)}{q^2} + F(w(q))].$$

As above, for d close enough to d_0 and for large enough q the expression in brackets is negative, and this is a contradiction. Thus, we must have

$$\zeta = \lim_{r\to\infty} w(r,d_0) = 0.$$

This completes the proof of the existence of a positive solution of (1.3)-(1.4) with $w(r) \to 0$ as $r \to \infty$.

Next, we define

$$A_1 = \{d > d_0 | w(r,d) \text{ has at most one positive zero.}\}$$

It follows from the definition of d_0 and Lemma 4.4 that A_1 is nonempty. From Lemmas 4.2 - 4.3 it follows that A_1 is bounded above. Thus, we define

$$d_1 = \sup A_1.$$

As above we can show that

$$w(r,d_1) \text{ has exactly one zero, and } w(r,d_1) \to 0 \text{ as } r \to \infty.$$

Proceeding inductively, we can find solutions that tend to zero at infinity and with any prescribed number of zeros. This completes the proof of the MAIN THEOREM.

6. EXPONENTIAL DECAY

Because $\lim_{s\to 0}\frac{f(s)}{s} = -\sigma^2$, we expect that solutions w of (1.3) that have $\lim_{r\to\infty} w(r) = 0$ will be governed for large r by the linearized equation

$$w'' + \frac{1}{r}w' - \frac{m^2}{r^2}w - \sigma^2 w = 0. \tag{6.1}$$

If $\sigma > 0$, equation (6.1) is the modified Bessel equation, whose decaying solution $K_m(\sigma r)$ has the asymptotic behavior

$$K_m(\sigma r) = \sqrt{\frac{\pi}{2\sigma}}\frac{1}{\sqrt{r}}e^{-\sigma r}(1 + O(\frac{1}{r})) \text{ as } r \to \infty.$$

Here we content ourselves with showing that

$$w(r) = O(e^{-\rho r}) \text{ as } r \to \infty \text{ for all } \rho \in (0, \sigma).$$

To do so, let $\sigma > 0$ and consider a solution to (1.3) - (1.4) such that $\lim_{r\to\infty} w(r) = 0$. The proof of the main theorem shows that there is an $r_0 > 0$ so large that w is monotonic for $r > r_0$. Without loss of generality we assume $w(r) > 0$ and $w'(r) < 0$ for all $r > r_0$. Then from (1.3) we have

$$w'' + f(w) = \frac{m^2}{r^2}w - \frac{1}{r}w' > 0 \text{ for all } r > r_0.$$

Because $\lim_{w\to 0}\frac{f(w)}{w} = -\sigma^2$, for every ρ with $0 < \rho < \sigma$, there is $\epsilon_\rho > 0$ such that $f(w) < -\rho^2 w$ for all w in $(0, \epsilon_\rho)$. Choose $r_\rho > r_0$ so large that $0 < w(r) < \epsilon_\rho$ for all $r \ge r_\rho$. Then $f(w(r)) < -\rho^2 w(r)$ for all $r \ge r_\rho$, so we have

$$w'' > -f(w) > \rho^2 w \text{ for all } r \ge r_\rho.$$

Multiplying this inequality by $w'(r) < 0$ gives

$$(w'^2)' < \rho^2 (w^2)'.$$

Because $w(r) \to 0$ and $w'(r) \to 0$ as $r \to \infty$, we may integrate each side of the inequality from r to ∞ to obtain

$$w'^2 < \rho^2 w^2 \text{ for } r \ge r_\rho.$$

Using the facts that $w > 0$ and $w' < 0$ for $r \ge r_\rho$, we take square roots, divide by w, and integrate from r_ρ to r to obtain

$$w(r) < w(r_\rho)e^{-\rho(r - r_\rho)} \text{ for } r \ge r_\rho,$$

which establishes that $w(r) = O(e^{-\rho r})$ as $r \to \infty$.

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